

AFFINE HOMOGENEOUS STRICTLY PSEUDOCONVEX HYPERSURFACES OF THE TYPE $(1/2, 0)$ IN \mathbb{C}^3

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Abstract

In this paper we denote a type of affine homogeneous real hypersurface of \mathbb{C}^3 and present a classification of homogeneous surfaces of the type $(1/2, 0)$. The result was obtained by reducing the classification problem mentioned above to the problem of solving a system of nonlinear (quadratic) equations. Solutions of such system represent all the Lie algebras corresponding to the homogeneous surfaces under consideration. Some details of integration procedure are discussed for obtained algebras.

1. Introduction

The description problem for real submanifolds of complex spaces (CR-manifolds) having "reach" symmetry groups, and in particular homogeneous manifolds, has been widely studied in recent years (see, for example [1]-[10]).

The first classification result for *holomorphically* homogeneous real hypersurfaces of 2-dimensional complex spaces (3-dimensional CR-manifolds) was given by Cartan in [1]. Beloshapka and Kossovskiy completely described 4-dimensional homogeneous CR-manifolds in [10].

However, so far only some partial classification results have been obtained for the next 5-dimensional case of homogeneous CR-manifolds.

Note that any analytic CR-manifold can be (locally) considered as the one embedded in some complex space \mathbb{C}^n of appropriate dimension. As another obvious remark we emphasize that most of the mentioned classification results for holomorphic homogeneity are closely related to the affine homogeneity property. This remark explains the exceptional interest to the affine homogeneity in the holomorphic geometry context.

The classification of *affine* homogeneous real hypersurfaces of \mathbb{C}^2 was obtained in [11]. In this paper we consider affine homogeneity property for the 5-dimensional real hypersurfaces of 3-dimensional complex space.

A real hypersurface M of \mathbb{C}^n (or \mathbb{R}^n) is called *affine homogeneous near a point* $P \in M$ if some Lie subgroup $G(M)$ of affine group $Aff(n, \mathbb{C})$ (or $Aff(n, \mathbb{R})$) acts transitively on M in a neighborhood of P .

There are several approaches to the homogeneity problem for real hypersurfaces in holomorphic as well as in the affine cases. One of them is the algebraic approach which is based on the study of canonical equations of the real analytic hypersurfaces. Below we discuss only strictly pseudo-convex (SPC) hypersurfaces (one can find the definition of the strictly pseudo-convexity in [17]).

It was proved in [13] that in a neighborhood of any point an equation of real analytic hypersurface $M \in \mathbb{C}^3$ can be written after some affine transformation in the form:

$$v = |z_1|^2 + |z_2|^2 + \varepsilon_1(z_1^2 + \bar{z}_1^2) + \varepsilon_2(z_2^2 + \bar{z}_2^2) + \sum_{k+l+2m \geq 3} F_{klm}(z, \bar{z})u^m. \quad (1.1)$$

Here z_1, z_2, w are complex coordinates in \mathbb{C}^3 , $u = \operatorname{Re} w, v = \operatorname{Im} w$;

F_{klm} is a polynomial of degree k in variable $z = (z_1, z_2)$, degree l in variable $\bar{z} = (\bar{z}_1, \bar{z}_2)$, and degree m in variable u .

Unordered pair $(\varepsilon_1, \varepsilon_2)$ of real nonnegative coefficients from (1.1) is the affine invariant of the hypersurface at a fixed point. In the affine homoge-

neous case this pair will be called a *type* of the homogeneous hypersurface.

Complete classification of affine homogeneous hypersurfaces of the type $(1/2, 1/2)$ in \mathbb{C}^3 was presented in [14] - [16]. The tubular hypersurfaces (tubes) $T = \Gamma + i\mathbb{R}^3$ over strictly convex affine homogeneous hypersurfaces $\Gamma \subset \mathbb{R}^3$ are the main examples of homogeneous CR-manifolds in this case. However, it was shown in [14] - [16] that a lot of homogeneous hypersurfaces of $(1/2, 1/2)$ -type cannot be affinely reduced to the tubes.

In addition, recall that a big family of homogeneous hypersurfaces of the type $(\varepsilon, 0)$ where $0 < \varepsilon \neq 1/2$ was presented in [13].

In this paper we consider affine homogeneous hypersurfaces of the type $(1/2, 0)$. Following the scheme introduced in [14], we reduce the classification problem for affine homogeneous hypersurfaces to the problem of solving a system of nonlinear equations and construct a list of affine homogeneous hypersurfaces of the type $(1/2, 0)$. This list is relatively long and (as the authors hope) complete. However, it is clear that any classification result containing many objects needs to be checked by multiple verifications.

2. The main theorem

Theorem. *The following hypersurfaces of the type $(1/2, 0)$ are affine homogeneous manifolds in \mathbb{C}^3 :*

$$v = 2x_1^2 + |z_2|^2; \quad (2.1)$$

$$v = \exp(x_1) + |z_2|^2, \quad (2.2)$$

$$v = -\ln(1 + x_1) + |z_2|^2 \quad (x_1 > -1), \quad (2.3)$$

$$v = \pm(1 + x_1)^\alpha + |z_2|^2 \quad (x_1 > -1), \alpha \in \mathbb{R} \setminus \{0, 1, 2\}, \quad (2.4)$$

$$v = (1 + x_1) \ln(1 + x_1) + |z_2|^2 \quad (x_1 > -1), \quad (2.5)$$

$$v^2 = |z_1|^2 + |z_2|^2 \quad (v \neq 0), \quad (2.6)$$

$$v = \frac{x_1^2}{1-x_2} + |z_2|^2 \quad (x_2 \neq 1), \quad (2.7)$$

$$v = |z_1|e^{B \arg z_1} + |z_2|^2 \quad (z_1 \neq 0), \quad B \in \mathbb{R}; \quad (2.8)$$

$$v = x_1^{1-\alpha}|z_2|^{2\alpha} \quad (x_1 \cdot |z_2| > 0), \quad \alpha \in \mathbb{R} \setminus \{0, 1\}; \quad (2.9)$$

$$Re(\bar{z}_1 w) = (Re(z_1 \bar{z}_2))^\alpha \quad (Re(z_1 \bar{z}_2) > 0), \quad \alpha \in (-\infty, 0). \quad (2.10)$$

Remark. The \pm sign in equation (2.4) depends on the parameter α . The required sign should ensure the positiveness of a coefficient under the term x_1^2 in a Taylor expansion of the right-hand side of (2.4).

The main theorem has a certain connection with the problem of (local) classification of holomorphically homogeneous hypersurfaces of \mathbb{C}^3 .

For example, hypersurfaces (2.1)-(2.5) are well-known tubular hypersurfaces (tubes); hypersurface (2.8) is locally holomorphically equivalent to a sphere in \mathbb{C}^3 ; hypersurfaces (2.6), (2.9) and (2.10) are holomorphically equivalent to a non-spherical tubes

$$v = \ln x_1 + A \ln x_2 \quad (A \neq 1)$$

with 7-dimensional groups of holomorphic transformations.

A cubic hypersurface (2.7) is the most interesting among the listed manifolds. It is an element of 1-parameter family of affine homogeneous hypersurfaces

$$v = \frac{x_1^2}{x_2} + (Ax_2^2 + y_2^2) \quad (A \in \mathbb{R}).$$

Note that any such surface has a type $(1/2, (A-1)/2(A+1))$ if $A \neq -1$. So the simple generalization of the example (2.7) gives a big family of different affine homogeneous hypersurfaces.

All the hypersurfaces listed in the main theorem were obtained by constructing and integrating a large number of 5-dimensional Lie algebras related to homogeneous hypersurfaces of the type $(1/2,0)$.

Any such algebra $g(M)$ can be considered as the algebra of affine vector fields tangent to homogeneous hypersurface M . The values of these fields at a point $P \in M$ cover all the tangent plane $T_P M$.

It can be shown that every algebra $g(M)$ has a matrix representation, so we associate with any vector field

$$\begin{aligned} Z = & (a_1 z_1 + a_2 z_2 + a_3 w + p) \frac{\partial}{\partial z_1} + \\ & + (b_1 z_1 + b_2 z_2 + a_3 w + s) \frac{\partial}{\partial z_2} + \\ & + (c_1 z_1 + c_2 z_2 + c_3 w + q) \frac{\partial}{\partial w} \end{aligned} \quad (2.11)$$

a matrix of the form

$$Z = \begin{pmatrix} a_1 & a_2 & a_3 & p \\ b_1 & b_2 & b_3 & s \\ c_1 & c_2 & c_3 & q \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.12)$$

Here the Lie bracket of two vector fields corresponds to the usual commutator for matrices

$$[Z_1, Z_2] = Z_1 Z_2 - Z_2 Z_1.$$

Let $g(M)$ be a matrix Lie algebra related to the affine homogeneous hypersurface $M \subset \mathbb{C}^3$ and the surface itself is defined by canonical equation (1.1). The fourth column of the matrix (2.12) corresponds to the translation component of vector field. That's why the real linear span of the fourth columns of the matrices (2.12) coincides with the tangent plane $T_0 M$ and has dimension 5. Hence, $\dim_{\mathbb{R}} g(M) \geq 5$.

Starting from the part 3 of this article, we consider only 5-dimensional

Lie algebras with the bases $(m_1, m_2, m_3, m_4 \in \mathbb{R})$

$$\begin{aligned}
E_1 &= \begin{pmatrix} A1_1 & A2_1 & A3_1 & 1 \\ B1_1 & B2_1 & B3_1 & 0 \\ 4i & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} A1_2 & A2_2 & A3_2 & i \\ B1_2 & B2_2 & B3_2 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_3 &= \begin{pmatrix} A1_3 & A2_3 & A3_3 & 0 \\ B1_3 & B2_3 & B3_3 & 1 \\ 0 & 2i & m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_4 &= \begin{pmatrix} A1_4 & A2_4 & A3_4 & 0 \\ B1_4 & B2_4 & B3_4 & i \\ 0 & 2 & m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} A1_5 & A2_5 & A3_5 & 0 \\ B1_5 & B2_5 & B3_5 & 0 \\ 0 & 0 & m_5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.13}$$

It can be shown (see [13], [14]) that for the elements c_1, c_2 of the matrices (2.12) the following restrictions hold

$$q \in \mathbb{R}, \quad c_1 = 2i(\bar{p} + 2\varepsilon_1 p), \quad c_2 = 2i(\bar{s} + 2\varepsilon_1 s).$$

We declare that for the hypersurfaces of the type $(1/2, 0)$ it is sufficient to consider only 5-dimensional algebras and that all algebras of higher dimensions can be reduced to 5-dimensional ones. Here we omit the proof of this statement but it will be a part of a future big article that authors prepare. We only add that for algebras $g(M)$ corresponding to homogeneous $(1/2, 0)$ -type surfaces (similarly to the surfaces from the paper [14]), the following upper dimension estimate holds

$$\dim_{\mathbb{R}} g(M) \leq 7. \tag{2.14}$$

The equality $\dim_{\mathbb{R}} g(M) = 7$ is satisfied only for unique (up to affine equivalence) affine homogeneous hypersurface of the type $(1/2, 0)$, namely

for the quadric

$$v = |z_1|^2 + |z_2|^2 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) = 2x_1^2 + |z_2|^2. \quad (2.15)$$

As a basis of the algebra of affine vector fields for this surface we can use, for example, the matrices

$$\begin{aligned} E_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ E_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ E_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.16)$$

The algebra with the basis (2.16) will be used below for simplification of cumbersome integration of some 5-dimensional algebras. However, the main difficulty in our article is to construct a complete list of such algebras related to affine homogeneous hypersurfaces.

To obtain this list one need to study a system of quadratic equations which means the bracket-closedness of the linear space with the basis (2.13). Element-wise rewriting of 10 such matrix equations gives us the system of 120 scalar ones.

Using the specific form of the basis (2.13) one can extract (like in [14]) some relatively simple subsystems from this big system. Their solutions can be obtained by consideration of a large number of partial cases using any

computer algebra system. On the final step of the described procedure we obtain the desired list of algebras. One can mention here [3] as one of the first papers studying the Lie algebras and homogeneity by means of computer mathematics. Note also the recent work [19] where the main results were obtained in the same way.

Remark. Note that complete classification of all the 5-dimensional algebras (see [18]) is well-known. However, we don't know the simple method to choose algebras related to homogeneous hypersurfaces from this complete list of algebras.

3. Lie algebras related to homogeneous hypersurfaces of a type (1/2, 0)

Here we give the bases of nine types of 5-dimensional algebras related to homogeneous hypersurfaces, that were obtained by the method mentioned above. All the parameters m_k, t_j using below are arbitrary real numbers.

$$E_1 = \begin{pmatrix} m_1 & 0 & 0 & 1 \\ 0 & m_1 + \frac{im_1 t_{16}}{m_2} & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.1)$$

$$E_2 = \begin{pmatrix} m_2 & 0 & 0 & i \\ 0 & m_2 + it_{16} & 0 & 0 \\ 0 & 0 & 2m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 2t_7 - 2m_1 - 2im_2 & 0 & i(m_1 + im_2)(m_1 - t_7) & 1 \\ 0 & t_7 + it_8 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.2)$$

$$E_2 = \begin{pmatrix} 2m_2 & 0 & 0 & i \\ 0 & m_2 + it_{16} & 0 & 0 \\ 0 & 0 & 2m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & -m_1 + t_7 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & i(m_1 - t_7) & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} -m_2(m_1 - t_7) & 0 & 0 & 0 \\ 0 & -\frac{1}{2}(m_1 - t_7)(m_2 + it_{16}) & 0 & 0 \\ 0 & 0 & -m_2(m_1 - t_7) & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} t_1 & 0 & -\frac{i}{2}(m_1 - t_7)(t_1 - 2t_7) & 1 \\ 0 & t_7 + it_8 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3)$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -(m_1 - t_7) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & i(m_1 - t_7) & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} -2m_1 - 2im_2 & 0 & im_1(m_1 + im_2) & 1 \\ 0 & 2im_2 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.4)$$

$$E_2 = \begin{pmatrix} 2m_2 & 0 & 0 & i \\ 0 & 2m_2 + it_{16} & 0 & 0 \\ 0 & 0 & 2m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -m_1 - im_2 & 0 & 0 \\ -2im_2 & 0 & -m_1m_2 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & im_1 - m_2 & 0 & 0 \\ 2m_2 & 0 & -im_1m_2 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} -m_1m_2 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}m_1(2m_2 + it_{16}) & 0 & 0 \\ 0 & 0 & -m_1m_2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} t_1 & 2m_3 - 2im_4 & -\frac{i}{2}(t_1 m_1 + 2m_3^2 + 2m_4^2) & 1 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 2m_3 & -m_1 & 0 & 0 \\ 0 & m_3 - im_4 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 2m_4 & im_1 & 0 & 0 \\ 0 & m_4 + im_3 & 0 & i \\ 0 & 2 & 2m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} -2m_1 & m_3 - im_4 & im_1^2 & 1 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

$$E_3 = \begin{pmatrix} \frac{3}{2}m_3 & -m_1 & \frac{i}{4}m_1 m_3 & 0 \\ 0 & m_3 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} \frac{3}{2}m_4 & im_1 & \frac{i}{4}m_1m_4 & 0 \\ 0 & m_4 & 0 & i \\ 0 & 2 & 2m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} t_1 & t_3 + it_4 & -\frac{i}{2}(t_1m_1 + t_4^2 + t_3^2) & 1 \\ 2t_3 - 2it_4 & 0 & im_1(-t_3 + it_4) & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.7)$$

$$E_2 = \begin{pmatrix} 0 & t_4 - it_3 & \frac{1}{2}(t_3^2 + t_4^2) & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} t_3 - it_4 & -m_1 & \frac{i}{2}m_1(-t_3 + it_4) & 0 \\ 0 & -2it_4 & \frac{i}{2}(-t_3 + it_4)(t_3 + it_4) & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} -t_4 - it_3 & im_1 & \frac{1}{2}m_1(-t_3 + it_4) & 0 \\ 0 & -2it_3 & \frac{1}{2}(t_3^2 + t_4^2) & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} -\frac{i}{2}(t_3^2 + t_4^2) & \frac{i}{2}m_1(t_3 + it_4) & -\frac{1}{4}m_1(t_3^2 + t_4^2) & 0 \\ 0 & -\frac{i}{2}(t_3^2 + t_4^2) & 0 & 0 \\ 0 & 0 & -\frac{i}{2}(t_3^2 + t_4^2) & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} -2m_1 & 0 & im_1^2 & 1 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.8)$$

$$E_3 = \begin{pmatrix} m_3 & -m_1 & \frac{i}{2}m_1m_3 & 0 \\ 0 & m_3 - im_4 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} m_4 & im_1 & \frac{i}{2}m_1m_4 & 0 \\ 0 & m_4 + im_3 & 0 & i \\ 0 & 2 & 2m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 3t_7 - 2m_1 & 0 & \frac{i}{2}(2m_1 - t_7)(m_1 - t_7) & 1 \\ 0 & t_7 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.9)$$

$$E_2 = \begin{pmatrix} m_2 & 0 & \frac{i}{2}m_2(m_1 - t_7) & i \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & 2m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} m_3 & t_7 - m_1 & \frac{i}{2}m_3(m_1 - t_7) & 0 \\ 0 & m_3 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} m_4 & i(m_1 - t_7) & \frac{i}{2}m_4(m_1 - t_7) & 0 \\ 0 & m_4 & 0 & i \\ 0 & 2 & 2m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} -\frac{1}{2}m_2(m_1 - t_7) & 0 & -\frac{i}{4}m_2(m_1 - t_7)^2 & 0 \\ 0 & -\frac{1}{2}m_2(m_1 - t_7) & 0 & 0 \\ 0 & 0 & -m_2(m_1 - t_7) & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4. Integration of Lie algebras

Note that integrating of a particular Lie algebra can give different (affine inequivalent) homogeneous hypersurfaces. Hence, the classifications results for homogeneous manifolds that were formulated in terms of Lie algebras (see, for example [2]) need to be specified.

In this section we give the results of integration of nine types of algebras in expanded form. As it was mentioned above, integration of algebras (i.e. constructing the homogeneous hypersurfaces related to these algebras) is a complicated procedure. So in the section 5 we outline shortly only some details of integration of the algebras under study.

Theorem 4.1. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.1), (3.8), (3.9) is affine equivalent to a quadric*

$$v = 2x_1^2 + |z_2|^2.$$

Theorem 4.2. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.2) is affine equivalent to one of the following hypersurfaces:*

$$1) \quad v = \exp(x_1) + |z_2|^2,$$

$$2) \quad v = |z_1|e^{B \arg z_1} + |z_2|^2, \quad B \in \mathbb{R};$$

Theorem 4.3. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.3) is affine equivalent to one of the following hypersurfaces:*

- 1) $v = 2x_1^2 + |z_2|^2,$
- 2) $v = \exp(x_1) + |z_2|^2,$
- 3) $v = -\ln(1 + x_1) + |z_2|^2,$
- 4) $v = (1 + x_1) \ln(1 + x_1) + |z_2|^2,$
- 5) $v = \pm(1 + x_1)^\alpha + |z_2|^2, \quad \alpha \in \mathbb{R} \setminus \{0, 1, 2\}.$

Theorem 4.4. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.4) is affine equivalent to one of the following hypersurfaces:*

- 1) $v = 2x_1^2 + |z_2|^2,$
- 2) $v^2 = |z_1|^2 + |z_2|^2.$

Theorem 4.5. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.5) is affine equivalent to one of the following hypersurfaces:*

- 1) $v = 2x_1^2 + |z_2|^2,$
- 2) $v = (1 + x_1) \ln(1 + x_1) + |z_2|^2,$
- 3) $v = x_1^{(1-\alpha)} |z_2|^{2\alpha}, \quad \alpha \in \mathbb{R} \setminus \{0, 1\}.$

Theorem 4.6. *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.6) is affine equivalent to one of the following hypersurfaces:*

- 1) $v = 2x_1^2 + |z_2|^2,$
- 2) $v = \frac{x_1^2}{1 - x_2} + 2|z_2|^2.$

Theorem 4.7 *Integral variety of a type $(1/2, 0)$ related to any of the algebras (3.7) is affine equivalent to one of the following hypersurfaces:*

$$1) \quad v = 2x_1^2 + |z_2|^2,$$

$$2) \quad v = -\ln(1 + x_1) + |z_2|^2,$$

$$3) \quad \operatorname{Re}(\bar{z}_1 w) = (\operatorname{Re}(z_1 \bar{z}_2))^\alpha, \quad \alpha \in (-\infty, 0).$$

It is obvious that combination of these seven theorems leads to the main theorem stated above.

5. Remarks on the proofs of the theorems

First of all we note the role played by 7-dimensional Lie algebra (2.16) in the following reasonings. This algebra corresponds to the quadric $v = 2x_1^2 + |z_2|^2$. Many of the algebras mentioned above can be reduced to 5-dimensional subalgebras of (2.16) by means of suitable matrix similarity. This argument leads to the simple proof of the theorem 4.1.

5.1 Proof of the theorem 4.1

Any algebra in the family (3.1) is just a subalgebra of the algebra (2.16). This means that integral manifolds of such subalgebras coincide with the analogous manifold related to (2.16), i.e. any of them is a quadric (2.15).

Algebras in the families (3.8) and (3.9) are reduced to the subalgebras of (2.16) by means of matrix similarities. For instance, in the case (3.8) this is

the similarity $g^* = C^{-1}gC$, with the matrix

$$C = \begin{pmatrix} 1 & 0 & i\lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.1)$$

where $\lambda = m_1/2$.

A new basis of (3.8) takes the form

$$\begin{aligned} E_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ E_3 &= \begin{pmatrix} m_3 & 0 & 0 & 0 \\ 0 & m_3 - im_4 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ E_4 &= \begin{pmatrix} m_4 & 0 & 0 & 0 \\ 0 & m_4 + im_3 & 0 & i \\ 0 & 2 & 2m_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5.2)$$

Analogously any algebra (3.9) also turns under the similarity with matrix (5.1) and $\lambda = (m_1 - t_7)/2$ into subalgebra of (2.16).

Theorem 4.1 is proved.

The scheme of the study of another algebras is more complicated. Here one need to work with the systems of partial differential equations.

The fact that the vector field Z is tangent to the homogeneous surface $M = \{\Phi(z, \bar{z}, u, v) = 0\}$ can be expressed in the form

$$Re(Z(\Phi)|_M) = 0. \quad (5.3)$$

So we have the system of five equations (5.3) related to the basis vector fields for each algebra under consideration. The defining function

$$\Phi(z, \bar{z}, u, v) = -v + F(z, \bar{z}, u)$$

of homogeneous surface determined by unknown function $F(z, \bar{z}, u)$ can be found from every such system (5.3).

5.2 Discussion of the proof of the theorem 4.2

As in the previous case, here we use the similarity with the matrix (5.1) and the same $\lambda = (m_1 - t_7)/2$. But we consider the sum $E_5 + \frac{m_1 - t_7}{2} E_2$ instead of the matrix E_5 .

Then the basis of the new algebra takes the form

$$E_1 = \begin{pmatrix} 2im_2 & 0 & 0 & 1 \\ 0 & t_7 + it_8 & 0 & 0 \\ 4i & 0 & 2t_7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.4)$$

$$E_2 = \begin{pmatrix} m_2 & 0 & 0 & i \\ 0 & m_2 + it_{16} & 0 & 0 \\ 0 & 0 & 2m_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that one can easily integrate this algebra due to the presence of the triple of "trivial" matrices E_3, E_4, E_5 in its basis. For instance, the

equation (5.3) related to the field E_5 , means the independence of the function $F(z, \bar{z}, u)$ on the variable $u = \operatorname{Re} w$ (the *rigidity* of the surface under consideration in terms of several dimensional complex analysis) in this case.

Another two simple equations are related to the fields E_3, E_4

$$\operatorname{Re} \left(\frac{\partial F}{\partial z_2} + 2i z_2 \frac{i}{2} \right) = 0, \quad \operatorname{Re} \left(i \frac{\partial F}{\partial z_2} + 2 z_2 \frac{i}{2} \right) = 0.$$

Hence,

$$\frac{\partial F}{\partial z_2} = \bar{z}_2, \quad \text{i.e.} \quad F = F(z, \bar{z}) = |z_2|^2 + H(z_1, \bar{z}_1), \quad (5.5)$$

where $H(z_1, \bar{z}_1)$ is an arbitrary real-valued function on the variables z_1, \bar{z}_1 .

Two remaining more complicated equations (5.3) related to the fields E_1, E_2 of the basis (5.2) have the form

$$\begin{aligned} (2m_2 y_1 + 1) \frac{\partial H}{\partial x_1} - 2m_2 x_1 \frac{\partial H}{\partial y_1} &= 2t_7 H, \\ 2m_2 x_1 \frac{\partial H}{\partial x_1} + (2m_2 y_1 + 1) \frac{\partial H}{\partial y_1} &= 2m_2 H. \end{aligned} \quad (5.6)$$

Solving of this system of equations leads to two cases associated with the possible values of the parameter m_2 .

In the first case when $m_2 = 0$, the general solution of the system (5.6) has the form

$$H = C \exp(2t_7 x_1) \quad (5.7)$$

with arbitrary constant C .

It remains to note that in the case $C = 0$ (as well as in the case $C \neq 0, t_7 = 0$) formula (5.7) defines a Levi degenerate surface, while the condition $C < 0$ leads to the surface with a non-degenerate indefinite Levi form. We study only strictly pseudo-convex homogeneous surfaces, so it is necessary to consider only positive values of the constant C and non-zero values of the parameter t_7 in formula (5.7). In this case the equation of the required affine-homogeneous surface corresponding to (5.7) can be written as

$$v = \exp(2t_7 x_1 + \ln C) + |z_2|^2,$$

or in the form (2.3) after the change of variable $x_1^* = 2t_7x_1 + \ln C$.

In the second case, i.e. if $m_2 \neq 0$, the formal solution of the system (5.6) depends on two real parameters. However, by reasonings similar to those of the case $m_2 = 0$, we can rewrite the solution in the form (2.8) so it depends on just one parameter.

5.3 Comments to the proof of the Theorem 4.3

Using the similarity with the matrix (5.1) and taking $\lambda = (m_1 - t_7)/2$ we can construct new algebras with bases

$$E_1 = \begin{pmatrix} r & 0 & 0 & 1 \\ 0 & t_7 + it_8 & 0 & 0 \\ 4i & 0 & 2t_7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.8)$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $r = t_1 + 2m_1 - 2t_7$.

By analogy with the arguments above, we concentrate our attention on the field E_1 . After straightforward simplifications associated with the other basic fields, the equation of a surface under consideration takes the form

$$F = |z_2|^2 + H(x_1). \quad (5.9)$$

Here $H(x_1)$ is an analytic function presenting a solution of ODE

$$(1 + rx_1)H'(x_1) = 2t_7H + 4x_1. \quad (5.10)$$

The form of solution of (5.10) depends on the parameters t_7 and r . Thus, when $t_7 = 0, r = 0$ we get

$$H(x_1) = 2x_1^2 + C,$$

and the equation of homogeneous surface has the form (2.1). If $t_7 = 0, r \neq 0$, the equation (2.2) appears. If $t_7 \neq 0$, but $r = 0$, we get the equation (2.3).

Finally, in the general case, when $t_7 \neq 0, r \neq 0$, there are two types of solutions of (5.10): when $2t_7/r = 1$ we have the equation (2.5) of the homogeneous surface and if $\alpha = 2t_7/r \neq 1$ we obtain the equation (2.4).

5.4 The main points of the proof of the Theorem 4.4

Here, as in the previous case, it is convenient to start with the similarity transformation with the matrix (5.1) where $\lambda = m_1/2$. As a result, the parameter m_1 is excluded from the basis of the algebra.

Thereby, the parameter m_2 will play the main role in the discussion. Value $m_2 = 0$ yields a family of 5-dimensional algebras, each of which is a subalgebra of (2.16). So the quadric $v = 2x_1^2 + |z_2|^2$ appears in Theorem 4.4.

If $m_2 \neq 0$, any algebra of the family (3.4) is similar to the algebra with basis

$$\begin{aligned} E_1 &= \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+i\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ E_3 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.11)$$

where $\xi = t_{16}/2m_2 \in \mathbb{R}$.

One remark concerning the upper left 2×2 matrix block of the matrices E_1, E_3, E_4 (responsible for the action of the group $G(M)$ in the complex space $\mathbb{C}_{(z_1, z_2)}^2$ tangent to discussed homogeneous surfaces) essentially helps in building an integral manifold of this algebra. These three 2×2 Pauli matrices present the basis of the well-known algebra $su(2)$. It is a Lie algebra for the group $SU(2)$ which indicates the occurrence of the expression $|z_1|^2 + |z_2|^2$ in equation of discussed homogeneous hypersurfaces. Thus the appearance of the equation (2.6) of affine-homogeneous surface can be naturally explained in this case.

Note that in [13] a family

$$v^t = |z_1|^2 + |z_2|^2, \quad t \in \mathbb{R} \setminus \{0, 1, 2\}. \quad (5.12)$$

of affine homogeneous real hypersurfaces was constructed.

The value of $t = 2$ that is prohibited by this formula, corresponds to the surface (2.6).

Remark. Affine transformation groups of all the surfaces (5.12) and (2.6) are 6-dimensional and can be easily written. One can prove that there are only two (up to affine transformations) second order surfaces $v = 2x_1^2 + |z_2|^2$ and $v^2 = |z_1|^2 + |z_2|^2$ that are affinely homogeneous manifolds of the type $(0, 1/2)$ and have "reach" (more than 5-dimensional) affine transformation groups.

5.5 Outline of the proof of the Theorem 4.5

First, consider the case $m_3 + im_4 = 0$.

The similarity with the matrix (5.1) and $\lambda = m_1/2$ reduces the bases of discussed algebras to the form (5.8) where $r = t_1 + m_1/2$ and conditions $t_7 = t_8 = 0$ are added.

From the discussions above, the quadric $v = 2x_1^2 + |z_2|^2$ appears when $r = 0$ and if $r \neq 0$ we get the logarithmic surface $v = -\ln(1 + x_1) + |z_2|^2$.

In the main case

$$m_3 + im_4 \neq 0 \quad (5.13)$$

the proof of the Theorem 4.5 is based on the two following statements.

Proposition 4.1 *If $m_3 + im_4 \neq 0$ then any of the algebras (3.5) is similar to the algebra with the basis*

$$E_1 = \begin{pmatrix} t_1 & 0 & -i(t_1 m_1 + 2m_3^2 + 2m_4^2)/2 & 0 \\ 0 & 0 & 0 & 0 \\ 4i & 0 & 2m_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.14)$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 4.2. *Integral variety of any algebra (5.14) is affine equivalent to a surface of the following family:*

$$v = x_1^{(1-\alpha)} |z_2|^{2\alpha}, \quad \alpha \in \mathbb{R} \setminus \{0, 1\}.$$

We give a brief comment to the proof of Proposition 4.2.

Due to the simple form of the matrices E_2, E_4, E_5 , the study of the system of five equations corresponding to basic fields of algebra (5.14) becomes significantly simpler. We need to solve just two equations

$$(t_1 x_1 + AG) \frac{\partial G}{\partial x_1} = 4x_1 + 2m_1 G, \quad x_1 \frac{\partial G}{\partial x_1} + s \frac{\partial F}{\partial s} = G \quad (5.15)$$

for an analytic function $G(x_1, s)$. Here

$$A = \frac{1}{2}(t_1 m_1 + 2m_3^2 + 2m_4^2), \quad s = |z_2|^2,$$

and the desired equation of the surfaces has the form $v = G(x_1, |z_2|^2)$.

The common solution of the second equation of (5.15) can be presented (outside the set $x_1 = 0$) in the form

$$G = x_1 H(s/x_1) \quad (5.16)$$

with an arbitrary analytic function H . Then the first equation in (5.15) takes the form of ODE

$$\xi(t_1 + AH)H' = AH^2 + BH - 4, \quad (5.17)$$

where

$$\xi = \frac{s}{x_1}, \quad B = (t_1 - 2m_1).$$

The following consideration of two cases corresponding to the equality or inequality to zero of a parameter A , leads to a common formula (2.9) for the unknown homogeneous surfaces but with different domains of variation of the parameter α .

Note that the formulas which connect the parameters of the homogeneous surfaces families have extremely cumbersome form in different representations of these families. This fact, known for a long time (see [3]), also appears in the proof of the Theorem 4.5.

5.6 Outline of the proof of the Theorem 4.6

Here it is also convenient, as in the proof of the Theorem 4.5, to consider two cases related to the parameter $m_3 + im_4$. In the first simple case $m_3 = m_4 = 0$ the standard similarity (5.1) with $\lambda = m_1/2$ transforms the basis (3.6) to the basis of 5-dimensional subalgebra of the algebra (2.16). Consequently, in this case, the desired homogeneous surface is the standard quadric (2.15).

In the general case $m_3 + im_4 \neq 0$, we note that there is a natural symmetry between m_3 and m_4 in algebras of the family (3.6). It is sufficient therefore to discuss only a situation

$$m_3 \neq 0. \quad (5.18)$$

The consideration of the dual case $m_4 \neq 0$ gives the result which is affine equivalent to the one obtained with the assumption (5.18).

For any of the algebras (3.6) the spectrum of the matrix

$$E_3 = \begin{pmatrix} \frac{3}{2}m_3 & -m_1 & \frac{i}{4}m_1m_3 & 0 \\ 0 & m_3 & 0 & 1 \\ 0 & 2i & 2m_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

is presented by the set $(\frac{3}{2}m_3, m_3, 2m_3, 0)$.

Taking into account the condition (5.18), we can diagonalize this matrix. Due to the similarity with the matrix

$$S = \begin{pmatrix} 1 & m_1 & im_1 & m_1 \\ 0 & m_3 & 0 & 2m_3 \\ 0 & -2i & 2 & -2i \\ 0 & 0 & 0 & -2m_3^2 \end{pmatrix}$$

consisting of the eigenvectors of E_3 , one can switch to a new algebra.

Using another similarity with diagonal matrix

$$D = \text{diag} \left(\frac{m_3^2 + m_4^2}{m_3}, \frac{m_3 + im_4}{m_3}, \frac{m_3^2 + m_4^2}{m_3^2}, 1 \right)$$

any of the algebras under consideration becomes the basis independent of the parameters m_3, m_4

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.19)$$

$$E_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system of partial differential equations corresponding to such algebra contains three equations (taking into account the trivial form of the matrices E_2 and E_5)

$$x_2 \frac{\partial F}{\partial x_1} = 2x_1, \quad 2 \frac{\partial F}{\partial y_2} = -y_2, \quad (5.20)$$

$$3x_1 \frac{\partial F}{\partial x_1} + 2x_2 \frac{\partial F}{\partial x_2} + 2y_2 \frac{\partial F}{\partial y_2} = 4F$$

with unknown function $F(x_1, x_2, y_2)$.

After step-by-step integration of these equations we obtain the equation of a homogeneous surface in the form

$$F = \frac{x_1^2}{x_2} - \frac{1}{4}y_2^2 + Cx_2^2, \quad (5.21)$$

where C is an arbitrary constant.

Tracing the movement of the origin (that belongs to the unknown homogeneous surface) under the intermediate transformations, we can determine the value of the constant $C = -1/4$ in the formula (5.21).

As a result, we arrive at equation (2.7) defining (up to affine transformation) a homogeneous surface in this case.

Remark. One can easily show that for $|C| \neq 1/4$ the surface (5.21) is affine homogeneous manifold of the type $(1/2, (1 + 4C)/(1 - 4C))$. All such (affinely different!) surfaces are the integral manifolds of the same algebra with basis (5.19). Note also that they are holomorphically equivalent to each other.

5.7 Outline of the proof of the Theorem 4.7

As in the previous discussion, we choose one complex parameter $t_3 + it_4$ from the real parameter quadruplets m_1, t_1, t_3, t_4 , on which the family (3.7) depends.

Note that the basis of the form (3.7) for $t_3 + it_4 = 0$ and the basis of the form (3.5) considered above identically coincide when $m_3 + im_4 = 0$. Consequently, the homogeneous surfaces obtained in these two subcases, i.e. $v = 2x_1^2 + |z_2|^2$ and $v = -\ln(1 + x_1) + |z_2|^2$ are the same in both cases.

Proposition 4.3. *If $t_3^2 + t_4^2 \neq 0$ then the basis of any algebra of the discussed family can be transformed by matrix similarity to the form*

$$E_1 = \begin{pmatrix} t_1 & -iA & 0 & 0 \\ 4i & 2m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.22)$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A = t_3^2 + t_4^2 + \frac{t_1 m_1}{2}$.

Outline of the proof of Proposition 4.3 is similar to the one from Proposition 4.1. In this case we have the symmetric dependence of the bases of the algebras on parameters t_3 and t_4 (similar to the parameters m_3 and m_4 in section 4.5). So, in fact it is sufficient to consider only the subcase $t_4 \neq 0$.

In this subcase at first we diagonalize (by the similarity) a matrix E_4 , and then consider linear combinations of obtained basis matrices. Such actions allow us to "improve" the basis in total.

At the next step of the proof of the Theorem 4.7 we simplify the upper left (2×2) -block e_1 of the matrix E_1 from the basis (5.22). If $A = t_3^2 + t_4^2 + \frac{t_1 m_1}{2}$ then $e_1 = \begin{pmatrix} t_1 & -iA \\ 4i & 2m_1 \end{pmatrix}$ has two real eigenvalues of opposite signs

$$\lambda_{1,2} = \frac{1}{2}((t_1 + 2m_1) \pm B), \quad \text{where } B = \sqrt{(t_1 + 2m_1)^2 + 16(t_3^2 + t_4^2)}. \quad (5.23)$$

Consequently, the matrix E_1 is diagonalizable and after an appropriate matrix similarity transformation it has the form

$$E_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha = \frac{\lambda_1}{\lambda_2} < 0.$$

We note here two important facts:

1) based on the formula (5.23), it is easy to show that the ratio α takes all the negative values depending on the parameters t_1, m_1, t_3, t_4 ;

2) it can be also shown that the similarity transformation diagonalizing E_1 does not actually change the rest of the basis matrices of the discussed algebras.

Now it remains to integrate algebras with the bases

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.24)$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 4.4. *Integral variety of any algebra with the basis (5.24) is affinely equivalent to one of the surfaces of the following family:*

$$Re(\bar{z}_1 w) = C(Re(z_1 \bar{z}_2))^\alpha, \quad \alpha \in (-\infty, 0), \quad C \in \mathbb{R}. \quad (5.25)$$

To prove the Proposition 4.4, at first we transform each basis matrix (5.24) into a lower triangular matrix that is convenient for the following integration. It can be easily done by the change of variables

$$z_1 = z_2^*, \quad z_2 = w^*, \quad w = z_1^* \quad (5.26)$$

(or, what is the same, by another similarity transformation).

After this transformation, the system of partial differential equations corresponding to the modified basic fields and to the desired equation $v = F(z, \bar{z}, u)$ of homogeneous surface, takes the form

$$E1 : x_2 \frac{\partial F}{\partial x_2} + y_2 \frac{\partial F}{\partial y_2} + \alpha(u \frac{\partial F}{\partial u} - F) = 0,$$

$$\begin{aligned}
E_2 : x_1 \frac{\partial F}{\partial x_2} + y_1 \frac{\partial F}{\partial y_2} &= 0, & E_3 : -y_1 \frac{\partial F}{\partial u} &= x_1, \\
E_4 : -(x_1 \frac{\partial F}{\partial x_1} + y_1 \frac{\partial F}{\partial y_1}) + (x_2 \frac{\partial F}{\partial x_2} + y_2 \frac{\partial F}{\partial y_2}) - (u \frac{\partial F}{\partial u} - F) &= 0 \\
E_5 : (-y_1 \frac{\partial F}{\partial x_1} + x_1 \frac{\partial F}{\partial y_1}) + (-y_2 \frac{\partial F}{\partial x_2} + x_2 \frac{\partial F}{\partial y_2}) - F \frac{\partial F}{\partial u} &= u.
\end{aligned} \tag{5.27}$$

Solving the equations corresponding to the fields E_3 , E_2 and E_4 sequentially, we obtain the formula

$$F = -\frac{x_1 u}{y_1} + \frac{1}{x_1} \cdot \varphi \left(\frac{y_1}{x_1}, x_1 y_2 - x_2 y_1 \right),$$

where φ is an arbitrary analytic function of two arguments.

Denoting

$$\xi = \frac{y_1}{x_1}, \quad r = x_1 y_2 - x_2 y_1,$$

we obtain the remaining two equations of (7.10) in the form

$$r \frac{\partial \varphi}{\partial r} = \alpha \varphi, \quad (\xi^2 + 1) \cdot \left(\frac{\partial \varphi}{\partial \xi} + \varphi \right) = 0. \tag{5.28}$$

The solving of this system leads us to the equation of the required homogeneous surface of the form

$$v + \frac{x_1 u}{y_1} = C \frac{1}{y_1} (x_1 y_2 - x_2 y_1)^\alpha$$

with an arbitrary real constant C .

In the complex variables, this equation can be rewritten as

$$Re(\bar{z}_1 w) = C (Im(\bar{z}_1 z_2))^\alpha.$$

The change of variable $z_2 = -iz_2^*$ let us use the symbol Re instead of the symbol Im in the right hand side of the equation. Proposition 4.4 is proved.

Note now that for zero constant C the last equation defines a degenerate surface in Levi sense. If C is a nonzero constant, we can assume that $C = 1$ due to the scaling of variable w . Thus, we prove the Theorem 4.7.

Remark. It is easy to verify directly that the equation (2.10) defines an affine homogeneous surfaces for all values of α . However, for $\alpha = 0$ the

surface (2.10) degenerates in the sense of Levi. Additional simple analysis of power series shows that for $\alpha > 0$ (2.10) describes the surfaces with an indefinite Levi form, that also do not satisfy the strictly pseudo-convexity condition.

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